UNIT-4

Systems with Two Degrees of Freedom

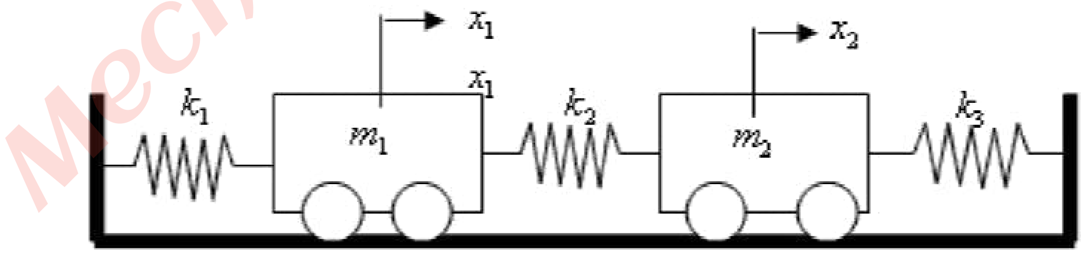
Introduction to two degree of freedom systems:

- The vibrating systems, which require two coordinates to describe its motion, are called two-degrees-of -freedom systems.
- These coordinates are called generalized coordinates when they are independent of each other and equal in number to the degrees of freedom of the system.
- Unlike single degree of freedom system, where only one co-ordinate and hence one equation of motion is required to express the vibration of the system, in two-dof systems minimum two co-ordinates and hence two equations of motion are required to represent the motion of the system. For a conservative natural system, these equations can be written by using mass and stiffness matrices.
- One may find a number of generalized co-ordinate systems to represent the motion of the same system. While using these co-ordinates the mass and stiffness matrices may be coupled or uncoupled. When the mass matrix is coupled, the system is said to be dynamically coupled and when the stiffness matrix is coupled, the system is known to be statically coupled.
- The set of co-ordinates for which both the mass and stiffness matrix are uncoupled, are known as principal co-ordinates. In this case both the system equations are independent and individually they can be solved as that of a single-dof system.
- A two-dof system differs from the single dof system in that it has two natural frequencies, and for each of the natural frequencies there corresponds a natural state of vibration with a displacement configuration known as the normal mode. Mathematical terms associated with these quantities are eigenvalues and eigenvectors
- Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed. A normal mode oscillation is defined as one in which each mass of the system undergoes harmonic motion of same frequency and passes the equilibrium position simultaneously.
- The study of two-dof- systems is important because one may extend the same concepts used in these cases to more than 2-dof- systems. Also in these cases one can easily obtain an analytical or closed-form solutions. But for more degrees of freedom systems numerical analysis using computer is required to find natural frequencies (eigenvalues) and mode shapes (eigenvectors).

The above points will be elaborated with the help of examples in this lecture.

Few examples of two-degree-of-freedom systems ::

Figure shows two masses m_1 and m_2 with three springs having spring stiffness k_1, k_2 and k_3 free to move on the horizontal surface. Let x_1 and x_2 be the displacement of mass respectively.



As described in the previous lectures one may easily derive the equation of motion by using d'Alembert principle or the energy principle (Lagrange principle or Hamilton 's principle)

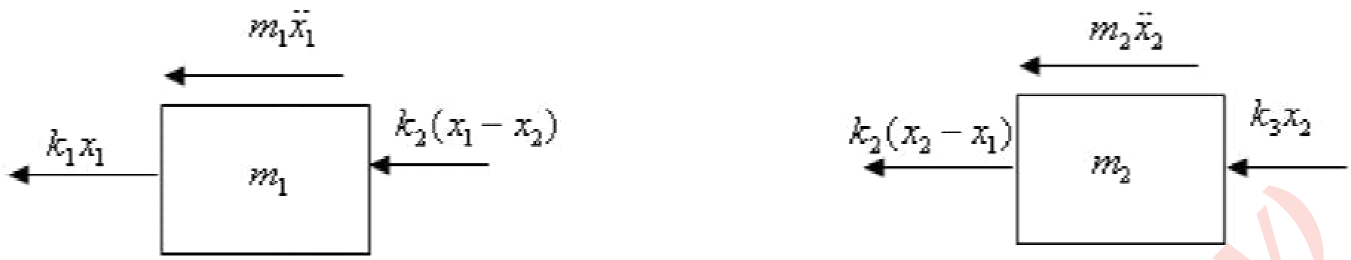


Figure 6.1.1(b): Free body diagrams

Using d'Alembert principle for mass m_1 from the free body diagram shown in figure.

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

and similarly for mass m_2

$$m_2\ddot{x}_2 - k_1 x_1 + (k_2 + k_3)x_2 = 0$$

Important points to remember

- Inertia force acts opposite to the direction of acceleration, so in both the free body diagrams inertia forces are shown towards left.
- For spring m_2 assuming $x_1 > x_2$, The spring will pull mass m_2 towards right by $k_2\,(x_2 - x_1)$ and it is stretched by $x_2 - x_1$ (towards right) it will exert a force of $k_2\,(x_2 - x_1)$ towards left on mass m_2 . Similarly assuming $x_1 > x_2$ the spring get compressed by an amount $x_2 - x_1$ and exert tensile force of $k_2\,(x_2 - x_1)$. One may note that in both cases, free body diagram remain unchanged.

Now if one uses Lagrange principle,

The Kinetic energy $$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

Potential energy $$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2$$

So, the Lagrangian

$$L = T - U = \left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2\right) - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2\right)$$

The equation of motion for this free vibration case can be found from the Lagrange principle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

and noting that the generalized co-ordinate $q_1 = x_1$ and $q_2 = x_2$

which yields

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_1x_1 + (k_2 + k_3)x_2 = 0$$

Same as obtained before using d'Alembert principle.

Now writing the equation of motion in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Here it may be noted that for the present two degree-of-freedom system, the system is dynamically uncoupled but statically coupled.

Free Vibration using normal modes: When the system is disturbed from its initial position, the resulting free-vibration of the system will be a combination of the different normal modes. The participation of different modes will depend on the initial conditions of displacements and velocities. So for a system the free vibration can be given by

$$x = \phi_1 A\sin(\omega_1 t + \psi_1) + \phi_2 B\sin(\omega_2 t + \psi_2)$$

Here *A* and *B* are part of participation of first and second modes respectively in the resulting free vibration and ψ_1 and ψ_2 are the phase difference. They depend on the initial conditions. This is explained with the help of the following example.

Forced harmonic Vibration:

Consider a system excited by a harmonic force $F_1 \sin \omega t$ expressed by the matrix equation

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \sin \omega t$$

Since the system is undamped, the solution can be assumed as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t$

Hence

$$\begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t = \begin{pmatrix} F_1 \\ 0 \end{pmatrix} \sin \omega t$$

Or $$\begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}^{-1} \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$= \frac{\begin{bmatrix} k_{22} - m_{22}\omega^2 & -k_{12} + m_{12}\omega^2 \\ -k_{21} + m_{21}\omega^2 & k_{11} - m_{11}\omega^2 \end{bmatrix} \begin{pmatrix} F \\ 0 \end{pmatrix}}{\begin{vmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{vmatrix}}$$

Hence

$$X_1 = \frac{\left(k_{22} - m_{22}\omega^2\right) F}{|Z(\omega)|}$$

Where $$[Z(\omega)] = \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}$$

$$X_2 = \frac{\left(k_{21} - m_{21}\omega^2\right) F}{|Z(\omega)|}$$

- Two vectors x_1 and x_2 are normal if $x_1^T x_1 = 1$ and $x_2^T x_2 = 1$
- Two vectors x_1 and x_2 are orthogonal if $x_1^T x_2 = 0$.

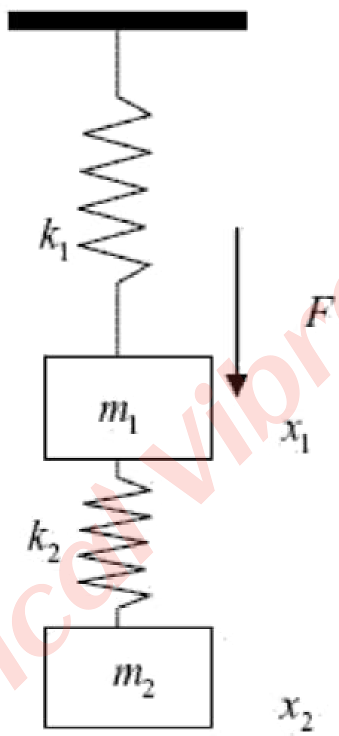
- If x_1 and x_2 normal and orthogonal, they are called orthonormal, in that case $x_i^T x_j = \delta_{ij} \quad i = 1,2, \quad j = 1,2$ where δ_{ij} is the Kronecker delta, defined by $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$

tuned Vibration Absorber

Consider a vibrating system of mass m_1, stiffness k_1, subjected to a force $F \sin \omega t$. As studied in case of forced vibration of single-degree of freedom system, the system will have a steady state response given by

$$x = \frac{F \sin \omega t}{m(\omega_n^2 - \omega^2)}, \text{ where } \omega_n = \sqrt{k_1 / m_1} \tag{1}$$

Which will be maximum when $\omega = \omega_n$. Now to absorb this vibration, one may add a secondary spring and mass system as shown in figure.



The equation of motion for this system can be given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \sin \omega t \\ 0 \end{pmatrix} \tag{2}$$

As we know for steady state vibration, the system will vibrate with a frequency of the external excitation; we can assume the solution to be

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t$$

(3)

Substituting Equation (3) in equation (2) one may write

$$\begin{pmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t = \begin{pmatrix} F \\ 0 \end{pmatrix} \sin \omega t$$

Or, $$\begin{pmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$ (5)

Using Cramer's rule one may write

$$X_1 = \frac{\left|\begin{pmatrix} F & -k_2 \\ 0 & k_2-m_2\omega^2 \end{pmatrix}\right|}{\left|\begin{pmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{pmatrix}\right|} = \frac{\left(k_2-m_2\omega^2\right)F}{|Z(\omega)|}$$

(6)

$$X_2 = \frac{\left|\begin{pmatrix} k_1+k_2-m_1\omega^2 & F \\ -k_2 & 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{pmatrix}\right|} = \frac{-k_2F}{|Z(\omega)|}$$

where $$Z(\omega) = \begin{pmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{pmatrix}$$ (9)

Now

$$|Z(\omega)| = \begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix} = k_1k_2 - m_1k_2\omega^2 - k_1m_2\omega^2 - k_2m_2\omega^2 + m_1m_2\omega^4$$

$$= m_1m_2(\lambda_1-\omega^2)(\lambda_2-\omega^2)$$ (10)

Here λ_1 and λ_2 are the roots of the characteristic equation $|Z(\omega)| = 0$. One may note that these roots are the normal mode frequency for this two-degrees of freedom system. These free-vibration frequencies can be given by

$$\lambda_{1,2} = 0.5\left\{\left(\frac{k_1}{m_1}+\frac{k_2}{m_2}+\frac{k_2}{m_1}\right) \pm \sqrt{\left(\frac{k_1}{m_1}+\frac{k_2}{m_2}+\frac{k_2}{m_1}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}}\right\}$$

From equation (6), it is clear that,

$$X_1 = 0, \text{ when } \left(k_2 - m_2\omega^2\right) = 0, \text{ or, when } \omega^2 = \frac{k_2}{m_2}.$$

Hence, if a system called the primary system with a stiffness k_1 mass m_1 is subjected to an exciting force or base motion to vibrate, it is possible to completely eliminate the vibration of the primary system by suitably designing an attached spring-mass system (secondary system) with stiffness k_2 and mass m_2 such that the natural frequency of the secondary system coincide with the exciting frequency.

$$\omega = \sqrt{\frac{k_2}{m_2}} \quad \textbf{(12)}$$

This is the principle of dynamic vibration absorber

From equation (1) it may be noted that the primary system will have resonance when the natural frequency of the primary system coincide with that of the excitation frequency.

Hence to reduce the vibration at resonance of the primary system one should design the secondary system such that the natural frequency of both the components coincides.

$$\omega_n^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1}$$

For this condition

$$X_1 = \frac{\left(k_2 - m_2\omega^2\right)F}{\left(k_1 + k_2 - m_1\omega^2\right)\left(k_2 - m_2\omega^2\right) - k_2^2}$$

$$= \frac{\dfrac{\left(k_2 - m_2\omega^2\right)F}{m_1 m_2}}{\left(\dfrac{k_1}{m_1}+\dfrac{k_2}{m_1}-\omega^2\right)\left(\dfrac{k_2}{m_2}-\omega^2\right) - \dfrac{k_2}{m_2}\dfrac{k_2}{m_1}}$$

Substituting $\mu = m_2 / m_1$ and $r = \omega / \omega_n$, the above equation reduces to

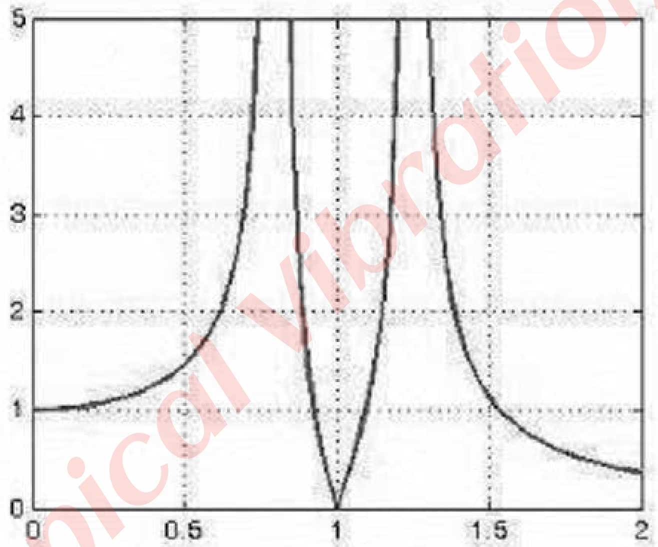
$$X_1 = \frac{\left(1-r^2\right)\left(F / k_1\right)}{\left(1+\mu-r^2\right)\left(1-r^2\right)-\mu}$$

$$\frac{k_1 X_1}{F} = \frac{\left(1-r^2\right)}{\left(1+\mu-r^2\right)\left(1-r^2\right)-\mu}$$

For, $\omega^2 = k_2 / m_2$,

$$|Z(\omega)| = k_1 k_2 - m_1 k_2 \frac{k_2}{m_2} - k_1 m_2 \frac{k_2}{m_2} - k_2 m_2 \frac{k_2}{m_2} + m_1 m_2 \frac{k_2}{m_2}\frac{k_2}{m_2}$$
$$= k_1 k_2 - m_1 \frac{k_2^2}{m_2} - k_1 k_2 - k_2^2 + m_1 \frac{k_2^2}{m_2} = -k_2^2$$

$X_1 = 0$ and $X_2 = \frac{-k_2 F}{-k_2^2} = \frac{F}{k_2}$



To keep the displacement of secondary mass small, the stiffness of the secondary spring should be very large. To have this the secondary mass should also be large which is not desirable from practical point of view.

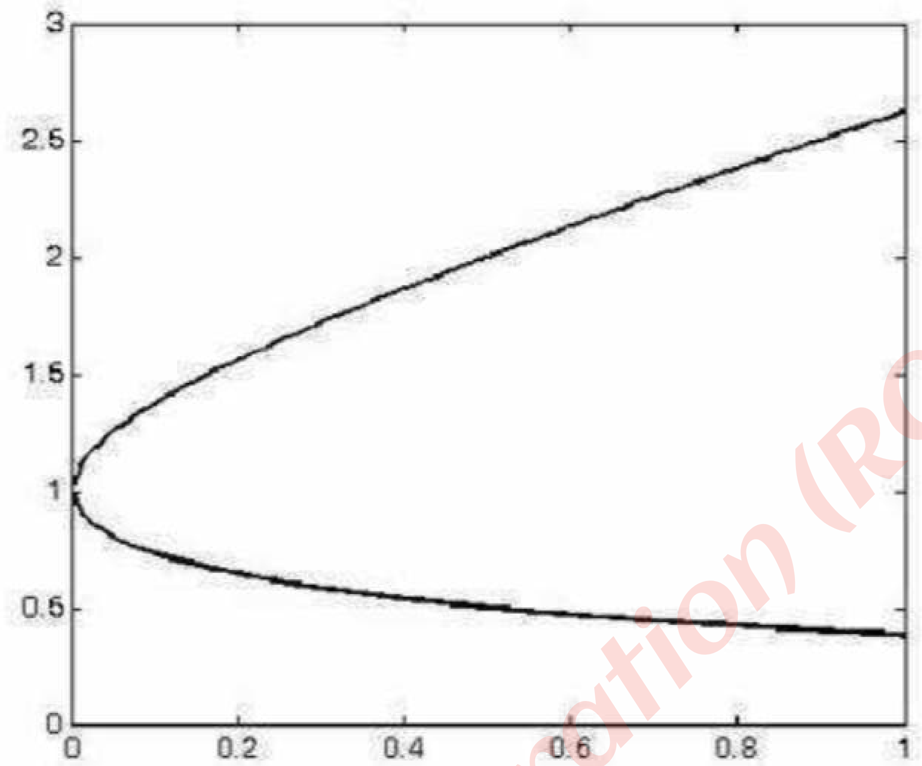
Hence a compromise is usually made between the amplitude and the mass ratio. The mass ratio is usually kept between 0.05 and 0.25.

Resonant frequency of the vibration absorber

$$\frac{\omega^4}{\omega_2^4}-(2+\mu)\left(\frac{\omega}{\omega_2}\right)^2+1=0$$

$$or, \left(\frac{\omega}{\omega_2}\right)^2=\left((2+\mu)\pm\sqrt{(2+\mu)^2-4}\right)/2$$

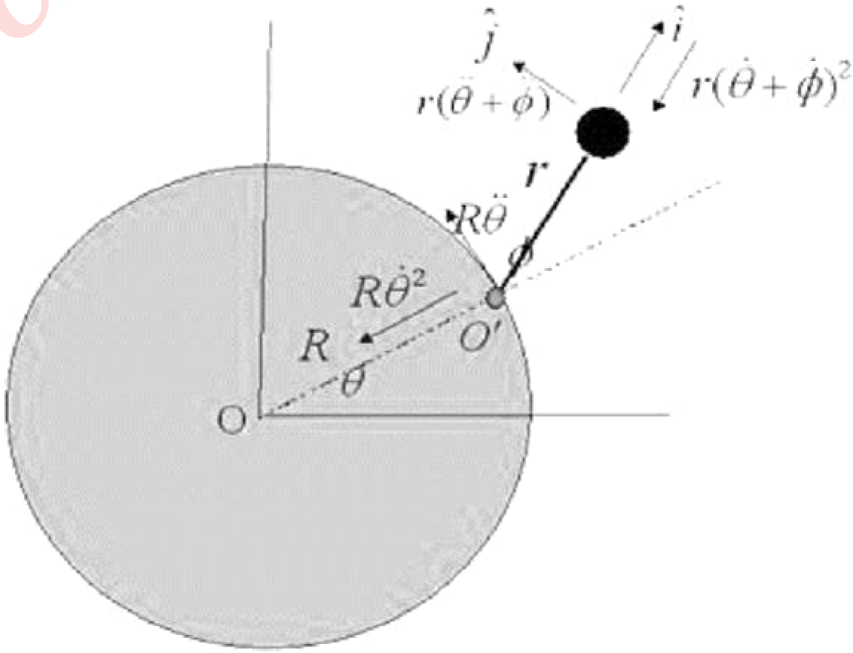
$$=\left(1+\frac{\mu}{2}\right)\pm\sqrt{\mu+\frac{\mu^2}{4}}$$



Centrifugal Pendulum Vibration Absorber

The centrifugal pendulum vibration absorber was devised and patented in France about 1935 and at the same time it was independently conceived and put into practice by E. S. Taylor. Its purpose was to overcome serious torsional vibration problem inherent in geared radial aircraft-engine –propeller system. Later it was modified and incorporated into automobile IC engines in order to reduce the torsional vibrations of the crankshaft. This was done by integrating the absorber mass with crankshaft counter balance mass.The tuned vibration absorber is only effective when the frequency of external excitation equals to the natural frequency of the secondary spring and mass system. But in many cases, for example in case of an automobile engine, the exciting torques are proportional to the rotational speed 'n' which may vary over a wide range. For the absorber to be effective, its natural frequency must also be proportional to the speed. The characteristics of the centrifugal pendulum are ideally suited for this purpose.

Placing the coordinates through point O', parallel and normal to r, the line r rotates with angular velocity ($\dot{\theta}+\dot{\phi}$).



The acceleration of mass m

$$a_m = \left[-R\dot{\theta}^2\cos\phi + R\ddot{\theta}\sin\phi - r(\dot{\theta}+\dot{\phi})^2\right]\hat{i} + \left[R\dot{\theta}^2\sin\phi + R\ddot{\theta}\cos\phi + r(\ddot{\theta}+\ddot{\phi})^2\right]\hat{j}$$

Since the moment about O' is zero,

$$M_{O'} = m\left[R\dot{\theta}^2\sin\phi + R\ddot{\theta}\cos\phi + r(\ddot{\theta}+\ddot{\phi})^2\right]r = 0$$

Assuming ϕ to be small, $\cos\phi = 1, \sin\phi = \phi$, so

$$\ddot{\phi} + \left(\frac{R}{r}\dot{\theta}^2\right)\phi = -\left(\frac{R+r}{r}\right)\ddot{\theta}$$

If we assume the motion of the wheel to be a steady rotation n plus a small sinusoidal oscillation of frequency ϖ, one may write

$$\theta = nt + \theta_0 \sin\varpi t$$

$$\dot{\theta} = nt + \varpi\theta_0\cos\varpi t \cong n$$

$$\ddot{\theta} = -\theta_0\varpi^2\sin\varpi t$$

Substituting the above equations in equation yields,

$$\ddot{\phi}+\left(\frac{R}{r}n^2\right)\phi=\left(\frac{R+r}{r}\right)\omega^2\theta_0\sin\omega t$$

Hence the natural frequency of the pendulum is

$$\omega_n=n\sqrt{\frac{R}{r}}$$

And its steady-state solution is

$$\phi=\frac{(R+r)/r}{-\omega^2+(Rn^2/r)}\omega^2\theta_0\sin\omega t$$

It may be noted that the same pendulum in a gravity field would have a natural frequency of $\sqrt{\frac{g}{r}}$. So it may be noted that for the centrifugal pendulum the gravity field is replaced by the centrifugal field Rn^2.