UNIT-3

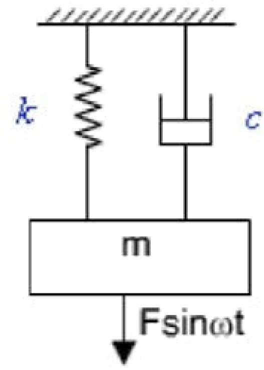
**Harmonically excited Vibration**

## Steady State Response due to Harmonic Oscillation:

Consider a spring-mass-damper system. The equation of motion of this system subjected to a harmonic force $F \sin \omega t$ can be given by

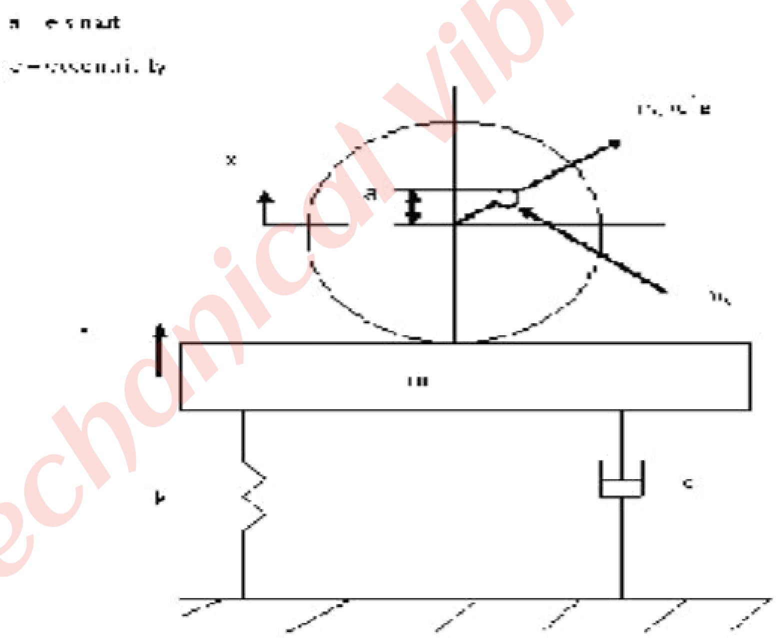
$$m\ddot{x} + kx + c\dot{x} = F \sin \omega t$$

Where, *m*, *k* and *c* are the mass, spring stiffness and damping coefficient of the system, *F* is the amplitude of the force, *w* is the excitation frequency or driving frequency.



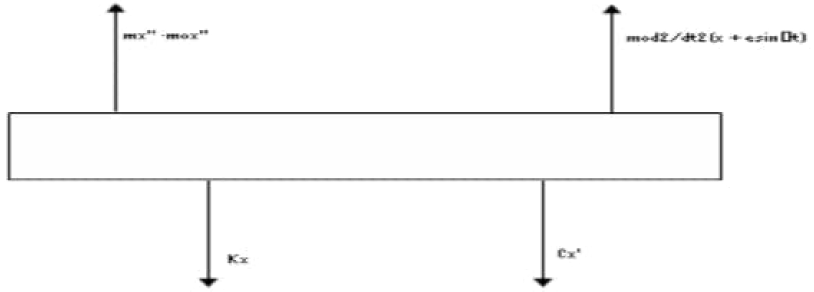

Alternative methods may be used to find the solution of equation

**Rotating and Reciprocating unbalance**



The figure shows a rotating equipment rotating at a speed of ω rad./sec. Let $m_o$ be the unbalance mass rotating with its CG at a distance of e from Centre. This unbalanced mass gives rise to a centrifugal force, equal to $m_o \omega^2 e$ .Let m be the total mass of equipment inclusive of $m_o$ and at any instant of time $m_o$ make an angle of ωt. The equation of motion for this system can be written considering the effective mass 'm-$m_o$' and the unbalanced mass '$m_o$'. Referring figure as shown below, we have the effective displacement of $m_o$ is sum of 'x' and 'esinωt'. Hence we can write the equation of motion in the vertical direction as

$$(m-m_o)x'' + (m_o)d^2(x + e\sin\omega t)/dt^2 = -Kx - Cx'$$




$$\text{i.e. } mx'' - m_o x'' + m_o x'' + m_o d\{\omega_e \cos\omega t\}/dt = -Kx - Cx'$$

$$mx'' - m_o \omega^2 e \sin\omega t = -Kx - Cx'$$

$$mx'' + Cx' + Kx = m_o\omega^2 \sin\omega t$$

The above equation is similar to

$$mx'' + Cx' + Kx = F\sin\omega t$$

Hence for an under damped system, we get the expression for steady state amplitude as

$$X = \frac{m_o\omega^2 e/K}{\sqrt{(1-(w/w_n)^2)^2 + (2lyw/w_n)^2}}$$

$$\text{Therefore} \quad \frac{X}{(m_o e/m)} = \frac{(w/wn)^2}{\sqrt{(1-(w/w_n)^2)^2 + (2lyw/w_n)^2}}$$

$\Phi = \tan^{-1}\{2\xi(w/w_n)/(1 - (w/w_n)^2)$

Same analysis is extended to reciprocating masses where exciting force becomes

$m_o ew^2 \sin wt$ where $m_o$ = Unbalanced mass of reciprocating masses.

The complete solution for the unbalanced system is

$$x = A_2 e^{-\xi\omega nt}(\sin\omega_d t + \Phi_2) + (m_o ew^2/k)/(\sqrt{(1-(\omega^2/\omega_n^2)^2 + (2\xi\omega/\omega_n)^2})$$

The following points are concluded for unbalanced system:

- Damping factor plays an important role in controlling the amplitudes during resonance.
- For low values of frequency ratio, X tends to 0.
- For low values of frequency ratio ($w/w_n$), X tends to 0.
- At high speeds of operation, damping effects are negligible.
- The peak amplitudes occur to right of resonance unlike for balanced systems.
- At resonance, $w = w_n$ ie: $X / m_o e/m = 1/2\xi$

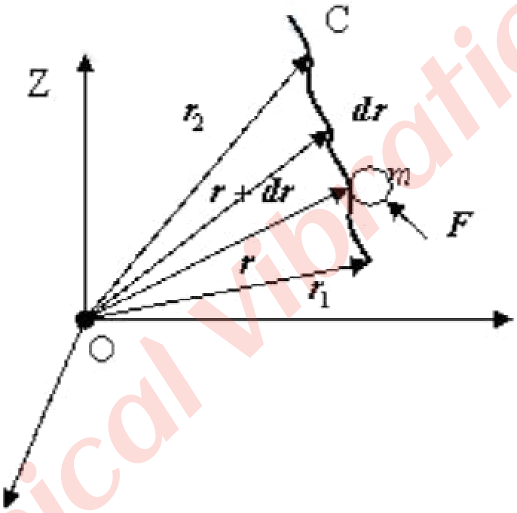
Also, $(X)_{resonance} = m_o e/ 2m\xi$

From the plot of $(X / em_o\, m)$ v/s $\omega/\omega_n$, it iserved that at low speeds, because the inertia force is small, all the curves start from zero and at resonance $(X / m_o e/m) = 1/2\xi$ and the amplitude of such vibrations can be controlled by the damping provided in the system. For very large frequency ratio, $(X / m_o e/m)$ tends to one.

## Newton's 2 nd law of the motion

*A particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector.*

Let a particle of mass $m$ moves along a curve C under the action of a given force *F* as shown in Figure 1. By definition the increment of work perform in moving the particle from position $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{dr}$ is given by




$$d\overline{W} = \boldsymbol{F}.\boldsymbol{dr}$$

The principle of virtual work is a statement of static equilibrium of mechanical systems which represents the first vibrational principle of mechanics and is a transition from Newtonian to Lagrangian mechanics.

### Virtual displacement

Consider a system of *N* particles in which the position of $i^{th}$ particle in space is represented by $r$ ( $i$ = 1,2... $N$ ). Then the virtual displacement represents the imagined infinitesimal changes $\delta r_i$ in these position vectors that are *consistent with the constraints of the system,* but are otherwise arbitrary.

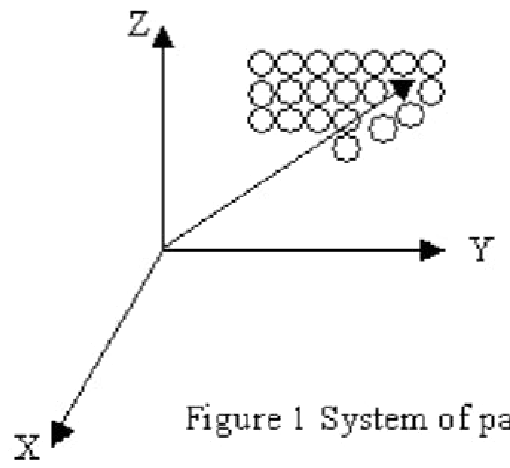
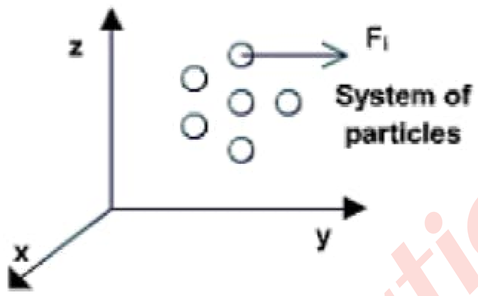



Figure 1 System of particles

According to the generalized d'Alembert principle the virtual work performed by the effective forces through infinitesimal virtual displacements compatible with the system constraints is zero.

$$\sum_{i=1}^{n}\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot \delta\vec{r}_i = 0 \qquad (1)$$




Consider the case where the displacements $r_i(t)$ are independent, so that $\delta r_i$ are entirely arbitrary. The virtual work done by the applied force can be given by -

$$\overline{\delta W} = \sum_{i=1}^{n}\vec{F}\cdot\delta\vec{r}_i \qquad (2)$$

### Forced Vibrations without Damping

The equation of motion of an undamped forced oscillator is:

$$mu'' + ku = F_0\cos(\omega t)$$

When $\omega \neq \omega_0$ (non-resonant case), the solution is of the form:

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m\left(\omega_0^{\,2} - \omega^2\right)}\cos(\omega t), \quad \omega_0 = \sqrt{\frac{k}{m}}$$

When $\omega = \omega_0$ (resonant case), the solution is of the form:

$$u(t)=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)+\frac{F_0}{2m\omega}t\sin(\omega_0 t)$$

## Forced Vibrations with Damping

In this section, we will restrict our discussion to the case where the forcing function is a sinusoid. Thus, we can make some general statements about the solution:

The equation of motion with damping will be given by:

$$mu''+\gamma u'+ku=F_0\cos(\omega t)$$

Its solution will be of the form:

$$u(t)=\underbrace{c_1u_1(t)+c_2u_2(t)}_{\substack{\text{homogeneous solution } u_h(t)\\ \text{"transient solution"}}}+\underbrace{A\cos(\omega t)+B\sin(\omega t)}_{\substack{\text{particular solution } u_p(t)\\ \text{"steady state solution"}}}$$

Notes:

- The homogeneous solution $u_h(t)\to 0$ as $t\to\infty$, which is why it is called the "transient solution."
- The constants c1 and c2 of the transient solution are used to satisfy given initial conditions.
- The particular solution $u_p(t)$ is all that remains after the transient solution dies away, and is a steady oscillation at the same frequency of the driving function. That is why it is called the "steady state solution," or the "forced response."
- The coefficients A and B must be determined by substitution into the differential equation.
- If we replace $u_p(t)=U(t)=A\cos(\omega t)+B\sin(\omega t)$ with $u_p(t)=U(t)=R\cos(\omega t-\delta)$, then $R=\frac{F_0}{\Delta}$, $\cos(\delta)=\frac{m(\omega_0^2-\omega^2)}{\Delta}$, $\sin(\delta)=\frac{\gamma\omega}{\Delta}$, $\Delta=\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$, and $\omega_0^2=\frac{k}{m}$. (See scanned notes at end for derivation)
- Note that as $\omega\to 0$, $\cos(\delta)\to 1$ and $\sin(\delta)\to 0\Rightarrow \boxed{\delta\to 0}$.

- Note that when $\omega = \omega_0$, $\boxed{\delta = \frac{\pi}{2}}$

- Note that as $\omega \to \infty$, $\boxed{\delta \to \pi}$ (mass is out of phase with drive).

- The amplitude of the steady state solution can be written as a function of all the parameters of the system:

$$R = \frac{F_0}{\Delta} = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}$$

$$= \frac{F_0}{\sqrt{m^2\omega_0^4\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \gamma^2\omega_0^2\frac{\omega^2}{\omega_0^2}}}$$

$$= \frac{F_0}{\sqrt{m^2\frac{k^2}{m^2}\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \gamma^2\frac{k}{m}\frac{\omega^2}{\omega_0^2}}}$$

$$= \frac{F_0}{k\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}}}$$

$$= \frac{F_0 / k}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}}, \Gamma = \frac{\gamma^2}{mk}$$

$$R\left(\frac{k}{F_0}\right) = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}}$$

- Notice that $R\left(\frac{k}{F_0}\right)$ is dimensionless (but proportional to the amplitude of the motion), since $\frac{F_0}{k}$ is the distance a force of $F_0$ would stretch a spring with spring constant *k*.

- Notice that $\Gamma = \frac{\gamma^2}{mk}$ is dimensionless... $\left[\frac{\gamma^2}{mk}\right] = \left[\frac{\left(\frac{mass}{time}\right)^2}{mass\frac{mass}{time^2}}\right] = [1]$

- Note that as $\omega \to 0$, $R\left(\frac{k}{F_0}\right) \to 1 \Rightarrow R \to \frac{F_0}{k}$.
- Note that as $\omega \to \infty$, $R \to 0$ (i.e., the drive is so fast that the system cannot respond to it and so it remains stationary).
- The frequency that generates the largest amplitude response is:

$$\frac{d}{d\omega}\left[R\left(\frac{k}{F_0}\right)\right] = 0$$

$$\frac{d}{d\omega}\left[\left(\left(1-\frac{\omega^2}{\omega_0{}^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0{}^2}\right)^{-\frac{1}{2}}\right] = \frac{-\frac{1}{2}\left(2\left(1-\frac{\omega^2}{\omega_0{}^2}\right)\left(\frac{-2\omega}{\omega_0{}^2}\right) + 2\Gamma\frac{\omega}{\omega_0{}^2}\right)}{\left(\left(1-\frac{\omega^2}{\omega_0{}^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0{}^2}\right)^{\frac{3}{2}}}$$

$$\left(\frac{\omega}{\omega_0{}^2}\right)\left(-2\left(1-\frac{\omega^2}{\omega_0{}^2}\right) + \Gamma\right) = 0$$

$$\omega = 0, \quad -2\left(1-\frac{\omega^2}{\omega_0{}^2}\right) + \Gamma = 0$$

$$1-\frac{\omega^2}{\omega_0{}^2} = \frac{\Gamma}{2}$$

$$\boxed{\omega_{\max}^2 = \omega_0{}^2\left(1+\frac{\Gamma}{2}\right)}$$

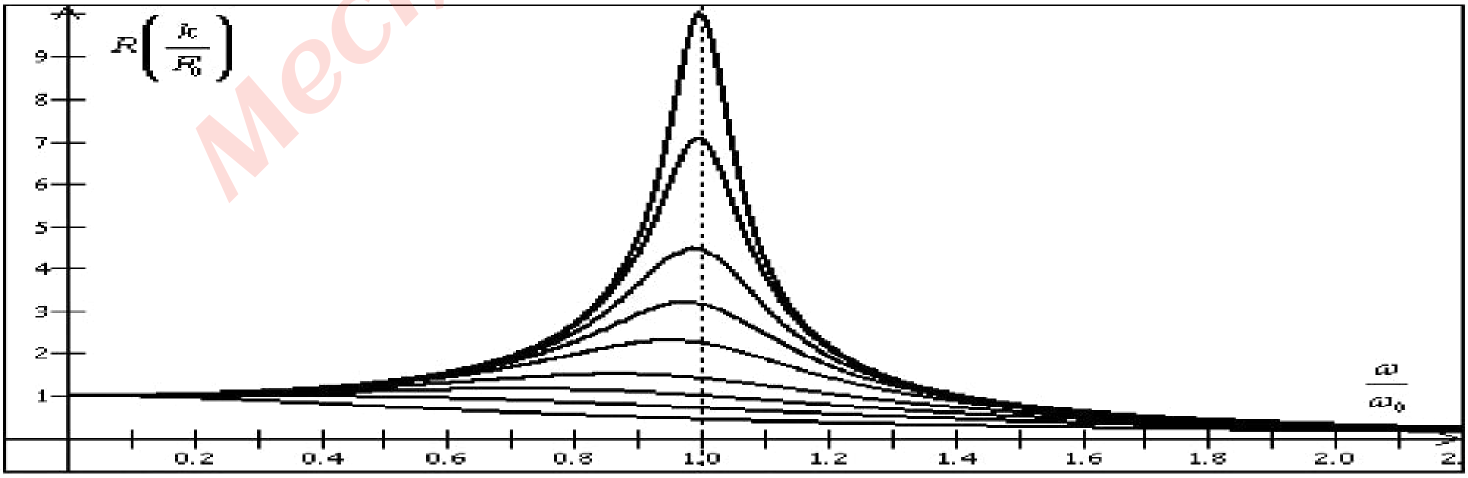
$$\omega_{\max}^2 = \omega_0{}^2 + \frac{k}{m}\frac{\gamma^2}{2mk}$$

$$\boxed{\omega_{\max}^2 = \omega_0{}^2 + \frac{\gamma^2}{2m^2}}$$

- Plugging this value of the frequency into the amplitude formula gives us:

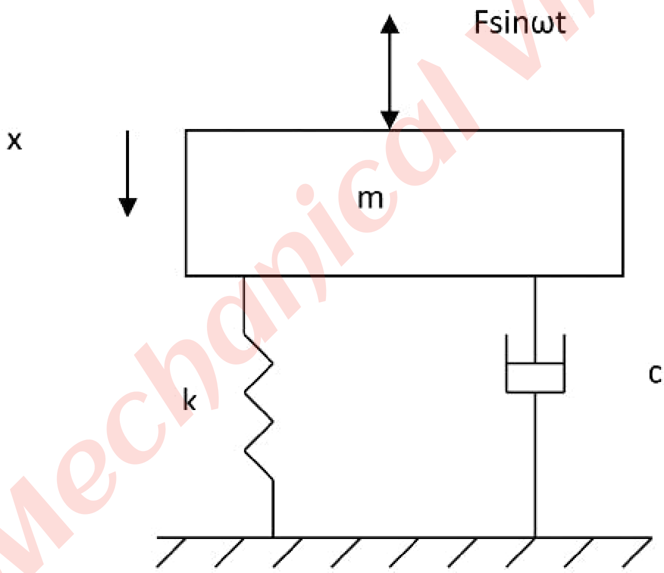
$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4mk}}}$$

- If $\frac{\gamma^2}{4mk} > 1$, then the maximum value of R occurs for $\omega = 0$.
- ***Resonance*** is the name for the phenomenon when the amplitude grows very large because the damping is relatively small and the drive frequency is close to the undriven frequency of oscillation of the system.




**Vibration Isolation:** High speed machines and engines due to unbalance give rise to vibrations of excessive amplitudes and due to the unbalance forces being setup, the foundations can be damaged. Hence there is a need to eliminate or reduce the vibrations being transmitted to the foundations, springs, dampers, etc. are placed between the machines and the foundations to reduce the vibrations or minimize then. These elements isolate the vibrations by absorbing the vibration energy. This isolation of vibrations is expressed in terms of force or motion transmitted to the foundation. The requirements of these isolating elements are that there should be no connection between the vibrating system & the foundation & it is to be ensured that in case of failure of isolators the system is still in of position on the foundation. Rubber acts effectively as an isolator during shear loading. The sound transmitted by it is also low. Heat and oil affect the rubber and it is usually preferred for light loads & high frequency oscillation. Felt pals are used for low frequency ratios. Many small sized felt pads are used instead of a single large pad. Cork can be used for compressive loads. Helical & leaf springs of metal are used as isolators for high frequency ratios. They are not affected by air, water or oil. The sound transmitted by them can be reduced by covering them with pads of felt, rubber or cork.

**TRANSMISSIBILITY:**




In a spring mass dashpot system subjected to harmonically varying external force, the spring and dashpot become the vibration isolators and the spring force and damping force are the forces between the mass and foundation. Thus the force transmitted to the foundation ($F_{tr}$) is vector sum of the spring force (kX) and damping force (cωX). We can write,

$F_{tr}= X \sqrt{(K^2 + c^2\omega^2)}$, substituting for X as $X = F/((k- m\omega^2)^2 + (c\omega)^2)$, we have $F_{tr}$ equal to,

$F_{tr} = F (\sqrt{(K^2 + c^2\omega^2)} / ((k- m\omega^2)^2 + (c\omega)^2)$

Transmissibility is defined as the ratio of force transmitted to the foundation to the force impressed on the system i.e.,

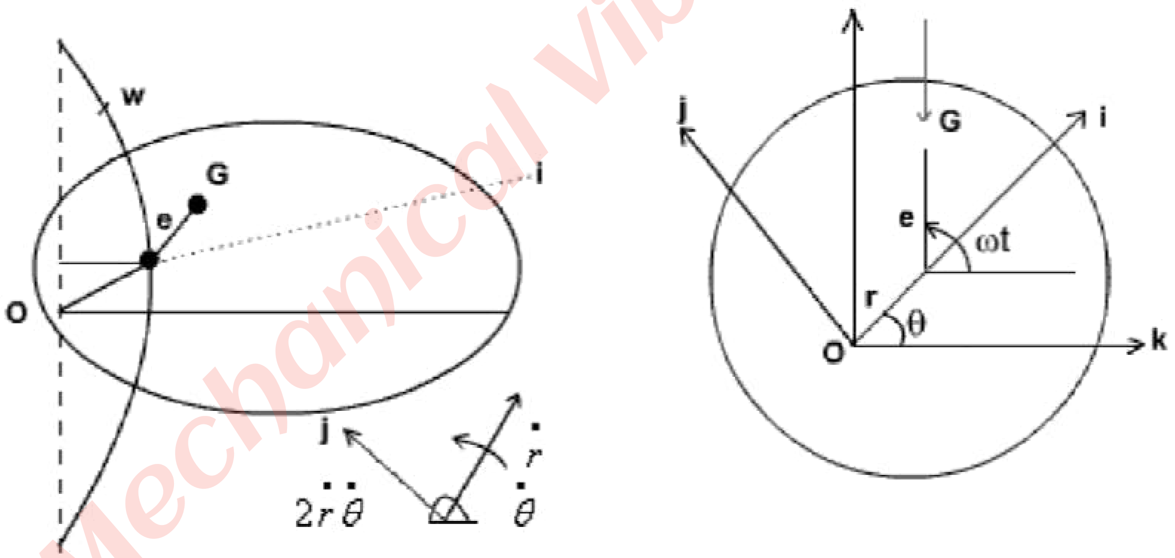
$T_r= \varepsilon = F_{tr}/ F = \sqrt{(1 + (c\omega/k)^2} / (\sqrt{(1-(\omega^2/\omega_n^2)^2} + (2\xi\omega/ \omega_n)^2 )$

The angle of lag of the transmitted force is ,

$(\phi - \alpha) = \tan^{-1}((2\xi\omega/ \omega_n) / 1-(\omega^2/\omega_n^2)) - \tan^{-1}(2\xi\omega/ \omega_n)$

Plot of $T_r$ versus $\omega/ \omega_n$ (refer a text book) for various values of ξ, is called the transmissibility curve. From the plot it is seen that all curves start from 1 and transmissibility $T_r$ is always desired to be less than 1, as it ensures that transmitted force to the foundation is minimum and better isolation is achieved. The operating values of frequency ratio to achieve this effect should be greater than √2 and the region beyond this value of frequency ratio is called mass control zone where isolation is most effective. In the plot the frequency ratio values up to 0.6 are spring control zone and from 0.6 to √2 is damping control zone and beyond that is mass control zone.

**Whirling of shaft:**
Whirling is defined as the rotation of the plane made by the bent shaft and the line of the centre of the bearing. It occurs due to a number of factors, some of which may include (i) eccentricity, (ii) unbalanced mass, (iii) gyroscopic forces, (iv) fluid friction in bearing, viscous damping.




Consider a shaft AB on which a disc is mounted at *S* . *G* is the mass center of the disc, which is at a distance *e* from *S*. As the mass center of the disc is not on the shaft center, when the shaft rotates, it will be subjected to a centrifugal force. This force will try to bend the shaft. Now the neutral axis of the shaft, which is represented by line ASB, is different from the line joining the bearing centers

AOB. The rotation of the plane containing the line joining bearing centers and the bend shaft (in this case it is AOBSA) is called the whirling of the shaft.

Considering unit vectors $\boldsymbol{i}$ , $\boldsymbol{j}$ , $\boldsymbol{k}$ as shown in the figure 4.6(b), the acceleration of point $G$ can be given by $a_G = a_S + a_{G/S}$

$$= \left[\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos\left(\omega t - \theta\right)\right]\boldsymbol{i} + \left[\boldsymbol{r}\ddot{\theta} - e\omega^2 \sin\left(\omega t - \theta\right) + 2\dot{r}\dot{\theta}\right]\boldsymbol{j}$$

Assuming a viscous damping acting at $S$. The equation of motion in radial direction

$$m\left[\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos\left(\omega t - \theta\right)\right] + kr + C\dot{r} = 0$$

$$m\left[r\theta + 2\dot{r}\dot{\theta} - e\ddot{\omega}^2 \sin\left(\omega t - \theta\right)\right] + cr\dot{\theta} = 0$$

$$\ddot{r} + \frac{c}{m}\dot{r} + \left(\frac{k}{m} - \dot{\theta}^2\right) = e\omega^2 \cos(\omega t - \theta)$$

$$r\ddot{\theta} + \left(\frac{c}{m}r + 2\dot{r}\right)\dot{\theta} = e\omega^2 \sin\left(\omega t - \theta\right)$$

Considering the synchronous whirl case, i.e. $\dot{\theta} = \omega$

$$\theta = \left(\omega t - \phi\right)$$

where $\phi$ is the phase angle between $e$ and $r$ .

Taking $\ddot{\theta} = \ddot{r} = \dot{r} = 0$ , from equation.

$$\left(\frac{k}{m} - \omega^2\right) = e\omega^2 \cos\phi$$

$$\frac{c}{m}r\omega = e\omega^2 \sin\phi$$

Hence, $$\tan\phi = \frac{\frac{c}{m}\omega}{\left(\frac{k}{m} - \omega^2\right)} = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

as $\omega_n = \sqrt{\dfrac{k}{m}}$ and $\zeta = \dfrac{c}{c_c}$

From equation, $$\cos\phi = \frac{\dfrac{c}{m}\omega}{\sqrt{\left(\dfrac{k}{m}-\omega^2\right)^2+\left(\dfrac{c}{m}\omega\right)^2}}$$

Substituting equation yields

$$\left(\frac{\mathrm{k}}{\mathrm{m}}-\omega^2\right)r = e\omega^2\frac{\left(\dfrac{k}{m}-\omega^2\right)}{\sqrt{\left(\dfrac{k}{m}-\omega^2\right)^2+\left(\dfrac{c}{m}\omega\right)^2}} = \frac{e\omega^2}{\sqrt{\left(\omega_n^2-\omega^2\right)+\left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}$$

or $$r = \frac{me\omega^2}{\sqrt{\left(k-m\omega^2\right)^2+\left(c\omega^2\right)^2}}$$

$$\frac{r}{e} = \frac{\omega^2/\omega_n^2}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\dfrac{\omega}{\omega_n}\right]^2}}$$

$$\tan\phi = \frac{2\zeta\dfrac{\omega}{\omega_n}}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}$$

The eccentricity line $e = SG$ leads the displacement line $r = OS$ by phase angle $\phi$ which depends on the amount of damping and the rotation speed ratio $\dfrac{\omega}{\omega_n}$ When the rotational speed equals to the natural frequency or critical speed, the amplitude is restrained by damping only. From equation at very high speed $\omega >> \omega_n$, $\phi \rightarrow 180^0$ and the center of mass $G$ tends to approach the fixed point $O$ and the shaft center S rotates about it in a circle of radius.