Unit -2

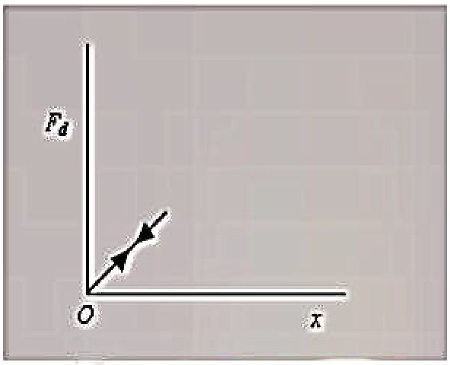
Damped Free Vibrations:

Damped System

Vibration systems may encounter damping of following types:

1. Internal molecular friction.
2. Sliding friction
3. Fluid resistance

Generally mathematical model of such damping is quite complicated and not suitable for vibration analysis.



Simplified mathematical model (such as viscous damping or dash-pot) have been developed which leads to simplified formulation. A mathematical model of damping in which force is proportional to displacement i.e., $F_d = cx$ is not possible because with cyclic motion this model will encounter an area of magnitude equal to zero as shown in figure So dissipation of energy is not possible with this model. The damping force (non-linearly related with displacement) versus displacement curve will enclose an area, it is referred as the hysteresis loop that is proportional to the energy lost per cycle.

Viscously damped free vibration:

Viscous damping force is expressed as,

$$F_d = c\dot{x}$$

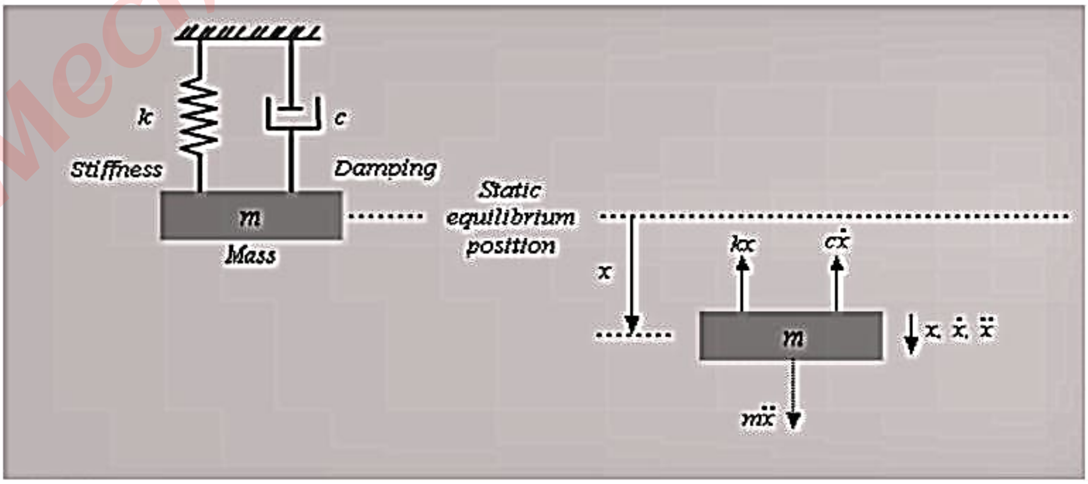
c is the constant of proportionality and it is called damping co-efficient.

From free body diagram, we have

$$\sum F = m\ddot{x}$$

$$-kx - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$



Let us assume a solution of equation of the following form

$$x = e^{st}$$

Where s is a constant (can be a complex number) and t is time. So that $\dot{x} = se^{st}$ and $\ddot{x} = s^2 e^{st}$, on substituting in equation we get,

$$(ms^2 + cs + k)e^{st} = 0$$

From the condition that equation is a solution for all values of t, above equation gives a characteristic equation (Frequency equation) as

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

has the following form

$ax^2 + bx + c = 0$ solution of which is given as $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Solution of equation (3.20) can be written as

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Hence the general solution is given by the equation

$$x = Ae^{s_1 t} + Be^{s_2 t}$$

Where A and B are integration constants to be determined from initial conditions.

$$x(t) = e^{-(c/2m)t}\left[Ae^{\sqrt{(c/2m)^2 - k/m}\,t} + Be^{-\sqrt{(c/2m)^2 - k/m}\,t}\right]$$

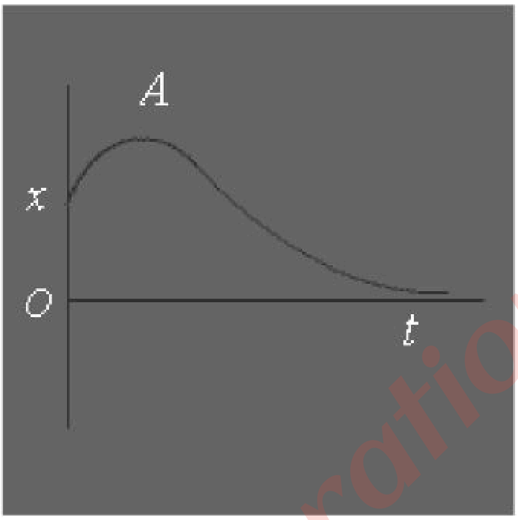
The term outside the bracket in RHS is an exponentially decaying function. The term

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Can have three cases.

$$\left(\frac{c}{2m}\right)^2 > \frac{k}{m}:$$

Exponents in will be real numbers.



Over damped system

$$\left(\frac{c}{2m}\right)^2 < \frac{k}{m}$$: exponents in are imaginary numbers : $$\pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

- we can write $$e^{\pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t} = \cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t \pm \sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t$$

Hence the takes the following form

$$x = e^{-\left(\frac{c}{2m}\right)t}\left[(A+B)\cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t + j(A-B)\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t\right]$$

$$e^{\pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t} = \cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t \pm \sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t$$

Let $a = (A+B) = X\cos\phi$ and $b = j(A-B) = X\sin\phi$, equation can be written as

$$x = Xe^{\left(\frac{c}{2m}\right)t}\cos\left[\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t-\phi\right]$$

where $\phi = \tan^{-1}(b/a)$; $X = \sqrt{a^2+b^2}$

(iii) Critical case between oscillatory and non-oscillatory motion: $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

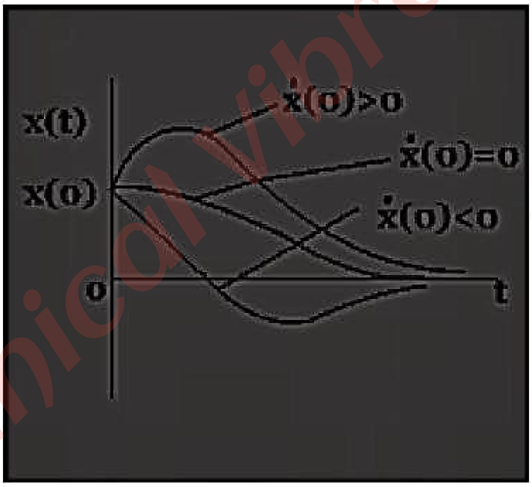
Damping corresponding to this case is called critical damping, c_c

$$c_c = 2m\sqrt{k/m} = 2m\omega_n = 2\sqrt{km}$$

Any damping can be expressed in terms of the critical damping by a non-dimensional number ζ called the damping ratio

$$\zeta = c/c_c$$

Response corresponding to the critical damping case is shown in for various initial conditions.



Critical damping

Equation of motion for damped system can be expressed in terms of ζ and ω_n as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

This form of equation is useful in identification of natural frequency and damping of system.

It is useful in modal summation of MDOF system also.

The roots of characteristic equation can be written as

$$s_{1,2} = \left[-\zeta \pm \sqrt{(\zeta^2 - 1)}\right]\omega_n$$

with $\frac{c}{m} = \frac{\zeta c_c}{m} = \frac{\zeta 2m\omega_n}{m} = 2\zeta\omega_n$

Depending upon value of damping ratio we can have the following cases

$\zeta > 1$, over damped condition

$\zeta < 1$, underdamped condition

$\zeta = 1$, critical damping

$\zeta = 0$, undamped system

Logarithmic Decrement:

Rate of decay of free vibration is a measure of damping present in a system. Greater is the decay, larger will be the damping.

Damped (free) vibration, general equation of the response is given as

$$x = \left(Xe^{-\zeta\omega_n t}\right)\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right)$$

Defining a term logarithmic decrement δ which is defined as the natural logarithm of the ratio of any two successive amplitudes as shown

$$\delta = \ln\frac{x_1}{x_2} = \ln\left[\frac{Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right)}{Xe^{-\zeta\omega_n(t_1+T_d)}\sin\left(\sqrt{1-\zeta^2}\,\omega_n\left(t_1+T_d\right)+\phi\right)}\right]$$

Since $\sin\left\{\sqrt{1-\zeta^2}\,\omega_n\left(t_1+T_d\right)\right\} = \sin\left\{\sqrt{1-\zeta^2}\,\omega_n t\right\}$

T_d = damped period, $T_d\omega_d = 2\pi$ where $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ = damped natural frequency

We have damped period $T_d = 2\pi/\omega_d = 2\pi/\omega_n\sqrt{1-\zeta^2}$, we get logarithmic decrement as

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

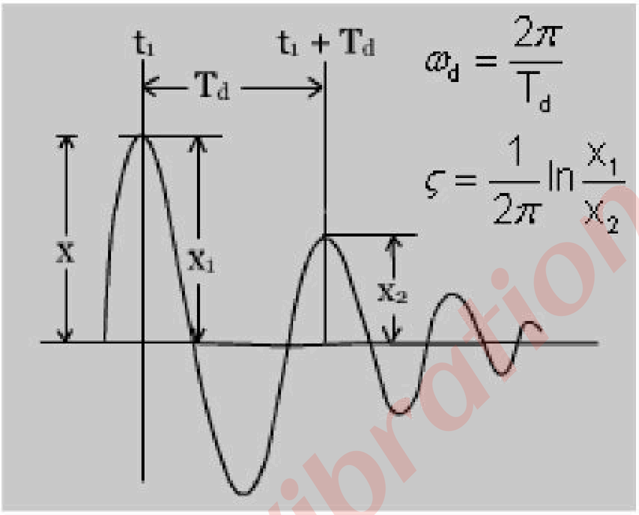
Since $\sqrt{1-\zeta^2} \cong 1$, the above equation reduces to

$$\delta \approx 2\pi\zeta$$

Experimental determination of natural frequency and damping ratio:

$\omega_d = \frac{2\pi}{T_d}$ rad/sec, T_d can be obtained from displacement-time free vibration oscillations.

$\zeta = \frac{1}{2\pi}\ln\frac{x_1}{x_2}$, where x_1 and x_2 are two consecutive amplitudes in the free vibration displacement-time curve.



The above illustration shows for two successive amplitude. But in case, the amplitude are recorded after "n" cycles, the formula is modified as

$$\frac{x_1}{x_2} = \frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot\frac{x_3}{x_4}\cdots\frac{x_{n-1}}{x_n}$$

Taking log,

$$\log_e\frac{x_1}{x_2} = \log\frac{x_1}{x_2} + \log\frac{x_2}{x_3} + \log\frac{x_3}{x_4} + \cdots + \log\frac{x_{n-1}}{x_n} = n\delta$$

$$\therefore \delta = \frac{1}{n}\log_e\frac{x_1}{x_n}$$

xample 3.1: A flywheel of weight 311.36 N is supported on a knife edge at distance 15.24 cm from its geometric center. If the measure period of oscillation was 1.22 sec, determine the mass moment of inertia of the flywheel about its geometric center.

(1) The flywheel can be considered as a compound pendulum. For a compound pendulum, the period is given by

$$T = 2\pi\sqrt{\frac{l+c}{g}}$$

T = Time period = 1.22 sec.

l = the distance between centers of rotation and gravity = 0.1524 m

g = acceleration due to 9.81 m/sec^2

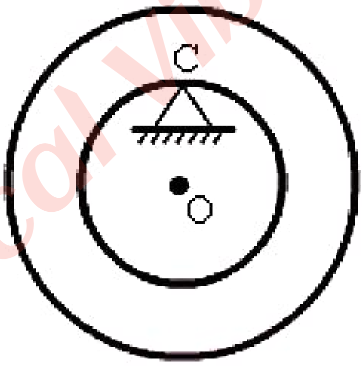
c = the distance between centers of gravity and percussion = $\frac{T^2 g}{4\pi^2} - l$

$$= \frac{(1.22)^2 \times 9.81}{4\times\pi^2} - 0.1524 = 0.3698 - 0.1524 = 0.2174 \text{ m}$$

Flywheel weight = 311.36 N

The mass moment of inertia of the flywheel about its geometric centre is given as

$$I_G = mK_G^2 = mlc = \frac{311.36}{9.81}\times 0.1524 \times 0.2174 = 10.315 \text{ kg m}^2$$



O - centre of gravity (or geometric centre)
C - centre of rotation or pivot

A flywheel pivoted on a knife edge